Numerical Solution of the Azimuthal-Invariant Thin-Layer Navier-Stokes Equations

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Abstract

NUMERICAL solutions have been obtained for a two-dimensional azimuthal- (or circumferentially) invariant form of the thin-layer Navier-Stokes equations. The governing equations which have been developed are generalized over the usual two-dimensional and axisymmetric formulation by allowing nonzero velocity components in the invariant direction. The equation formulation along with the solution method is described, and results for spinning and nonspinning bodies are presented.

Contents

The three-dimensional flowfield equations are frequently simplified for flowfields which are invariant in one coordinate direction. In the usual axisymmetric approximation, the azimuthal velocity is assumed to be zero, and the momentum equation in that direction can be eliminated. Thus, only four equations are required to be solved for four unknowns. However, for a variety of interesting flowfields, the velocity component in the invariant direction (here taken as η) is not zero although the governing equations are still two-dimensional. Examples include viscous flow about an infinitely swept wing, the viscous flow about a spinning axisymmetric body at 0-deg angle of attack, and axisymmetric swirl flows. Each of these flows can be solved as a two-dimensional problem although all three momentum equations have to be retained, and source terms replace the derivative of the flux terms in the η-direction.

Azimuthal-invariant equations are obtained from the three-dimensional equations[1] by making use of two restrictions: 1) all body geometries are of axisymmetric types and 2) the state variables and the contravariant velocities do not vary in the azimuthal direction. Here, η is used for the azimuthal coordinate, and the terms azimuthal and η-invariant will be used interchangeably. A sketch of a typical axisymmetric body is shown in Fig. 1a. In order to determine the circumferential variation of typical flow and geometric parameters, we first establish correspondence between the inertial Cartesian coordinates (x,y,z) (to which the dependent variables are referenced), the natural inertial cylindrical coordinates (x,ϕ,R), and the transformed variables (ξ,η,ζ). The choice of the independent variables $\xi,\eta,$ and ζ is restricted, as shown in Fig. 1c, insofar as η must vary as ϕ, i.e., $\phi=\mathcal{C}\eta$ (where $\mathcal{C}$ is a constant). From the views shown in Fig. 1, the relationship between the coordinate systems is observed to be: $y=R\ (\xi,\zeta,\tau)\sin\phi$, $\phi=\mathcal{C}\eta$, $z=R(\xi,\zeta,\tau)\ \cos\phi$, $x=x(\xi,\zeta,\tau)$ where $\phi=\phi(\tau)$, and the Cartesian and cylindrical coordinates are related in the usual way. Note that x and R are general functions of only ξ,ζ, and τ.

Using the relationships given in Eq. (1), the thin-layer azimuthal-invariant equations written in conservation law form are: $\partial_\tau\hat{q}+\partial_\xi\hat{E}+\partial_\zeta\hat{G}+\hat{H}=Re^{-1}\partial_\zeta\hat{S}$ where the general

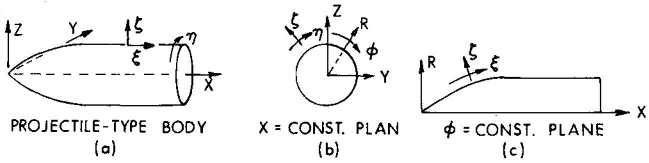

Fig. 1 Axisymmetric body and coordinate systems.

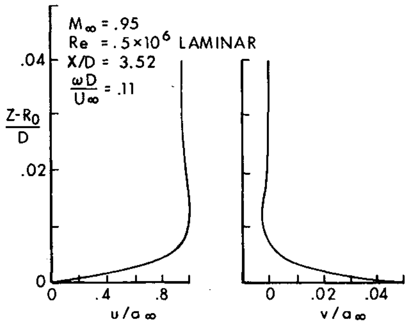

Fig. 2 Velocity profile for spinning projectile.

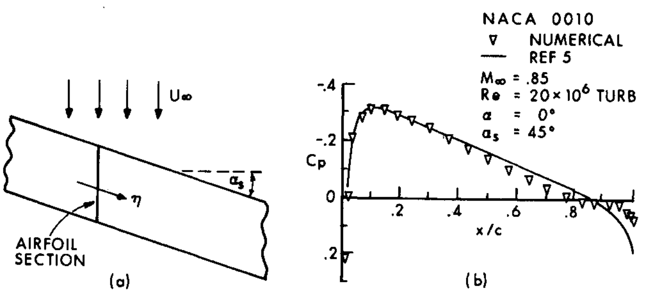

Fig. 3 Infinite swept wing.

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coordinate transformations

$$\xi=\xi(x,y,z,t),\quad \zeta=\zeta(x,y,z,t),\quad \eta=\eta(y,z,t)\quad \tau=t$$

are used, and

$$\hat{q}=J^{-1}\begin{vmatrix}\rho\\ \rho u\\ \rho v\\ \rho w\\ e\end{vmatrix}\quad \hat{E}=J^{-1}\begin{vmatrix}\rho U\\ \rho uU+\xi_x p\\ \rho vU\\ \rho wU+\xi_z p\\ (e+p)U-\xi_t p\end{vmatrix}\quad \hat{G}=J^{-1}\begin{vmatrix}\rho W\\ \rho uW+\zeta_x p\\ \rho vW\\ \rho wW+\zeta_z p\\ (e+p)W-\eta_t p\end{vmatrix}\quad \hat{H}=J^{-1}\phi_\eta\begin{vmatrix}0\\ 0\\ \rho V[R_\xi(U-\xi_t)+R_\zeta(W-\zeta_t)]\\ -\rho VR\phi_\eta(V-\eta_t)-p/(R\phi_\eta)\\ 0\end{vmatrix}$$

The vector $\hat{q}$ contains the five dependent variables: density ρ; three velocity components $u,v,$ and w; and total energy e. Here, J is the transform Jacobian. The thin-layer viscous term $\hat{S}$ is given in Refs. 1 and 2. Further details of the development of the η-invariant equations can be found in the full paper.[2]

Note this formulation contains only two spatial derivatives but does retain all three momentum equations, thus allowing a degree of generality over the standard axisymmetric equations. Additionally, only viscous terms in the ζ-direction have been retained. This so-called "thin-layer" approximation[1,3,4] requires that all body surfaces be mapped onto ζ = constant planes and that $Re \gg 1$. Essentially, all the viscous terms in the coordinate directions (here taken as ξ and η) along the body surface are neglected, while terms in the ζ or the near-normal direction to the body are retained.

The numerical algorithm used is a fully implicit, approximately factored, finite-difference algorithm written in delta form as analyzed by Beam and Warming.[5] The finite-difference algorithm can be first- or second-order accurate in time and second- or fourth-order accurate in space. The solution of the two-dimensional system of difference equations is implemented by an approximate factorization of the equations into two one-dimensional-like systems of equations. This procedure has been utilized in previous applications[1,3-5] similar to ours, and the bibliographies of Refs. 1-5 should be referred to for numerous related works.

The η-invariant equations have been applied to a number of axisymmetric and planar type flow problems. In particular, the η-invariant code has been verified by computing conventional axisymmetric flows. The code has additionally been applied to hollow projectiles (i.e., ring airfoils), the results for which can be found in the full paper. This application was performed to show the versatility of the numerical algorithm with its general geometry capability.

A viscous solution was obtained for a nonspinning projectile with a 3-caliber (i.e., 3 max body diam) ogive nose, 2-caliber centerbody, and 1-caliber 7-deg boattail at $M_\infty = 0.96$ and zero angle of attack. The calculated pressure coefficient compared well with the experimental data and is presented in the full paper. The grid for these calculations contained 80 longitudinal points in ξ with clustering at the boattail, and 50 points in the radial (ζ) direction with exponential stretching away from the body.

A major motivation for the development of the η-invariant code was to obtain the capability to compute flowfields about spinning projectile shapes. Results presented in Fig. 2 show the boundary-layer profiles obtained for a similar projectile when spun at a nondimensional spin parameter of $\omega D/U_\infty = 0.11$, where ω is the angular velocity, and D is the maximum body diameter. The surface pressure distributions were practically identical for the spinning and nonspinning cases, and even the streamwise velocity profile was essentially unaffected by the addition of this amount of spin. The v component of velocity is shown to decrease from a maximum value at the surface to effectively zero at the edge of the boundary layer.

The flow about an infinitely swept wing is another example of an η-invariant flow where, here, the η coordinate is chosen parallel to the wing leading edge, as shown in Fig. 3a. For inviscid flow, one can use simple sweep theory to convert this problem into a conventional two-dimensional airflow problem. In viscous flow, however, one finds that because the streamlines over the wing are curved, the crossflow momentum equation must be retained.

The η-invariant equations for an infinitely swept wing are most easily obtained by considering this problem as a limiting case of the hollow projectile. From this point of view, we consider the flow about a ring airfoil with $R \to \infty$ and $\phi_\eta \to 0$. As $R \to \infty$ and $\phi_\eta \to 0$, the term $R\phi_\eta \to 1$ and the entire source term $H \to 0$. A turbulent calculation at $Re = 20\times 10^6$ about a NACA 0010 cross-sectional wing with 45-deg sweep, $\alpha = 0$ deg, and $M_\infty = 0.85$ is compared with an inviscid small perturbation solution obtained with the Bailey-Ballhaus[6] three-dimensional wing program, as shown in Fig. 4b. For this case, viscous effects are somewhat mild, but even so, a discrepancy between the inviscid and viscous solution is evident from the trailing edge up to approximately the midchord.

References

[1] Pulliam, T. H. and Steger, J. L., "On Implicit Finite-Difference Simulations of Three-Dimensional Flow," AIAA Paper 78-10, Jan. 1978.

[2] Nietubicz, C. J., Pulliam, T. H., and Steger, J. L., "Numerical Solution of the Azimuthal-Invariant Thin-Layer Navier-Stokes Equations," AIAA Paper 79-0010, Jan. 1979.

[3] Steger, J. L., "Implicit Finite-Difference Simulation of Flow about Arbitrary Geometries with Application to Airfoils," AIAA Paper 77-665, June 1977.

[4] Baldwin, B. S. and Lomax, H., "Thin-Layer Approximation and Algebraic Model for Separated Turbulent Flows," AIAA Paper 78-257, Jan. 1978.

[5] Beam, R. and Warming, R. F., "An Implicit Factored Scheme for the Compressible Navier-Stokes Equations," AIAA Paper 77-645, June 1977.

[6] Bailey, F. R. and Ballhaus, W. F., "Comparison of Computed and Experimental Pressures for Transonic Flow about Isolated Wings and Wing-Fuselage Configurations," NASA CP-347, March 1975.